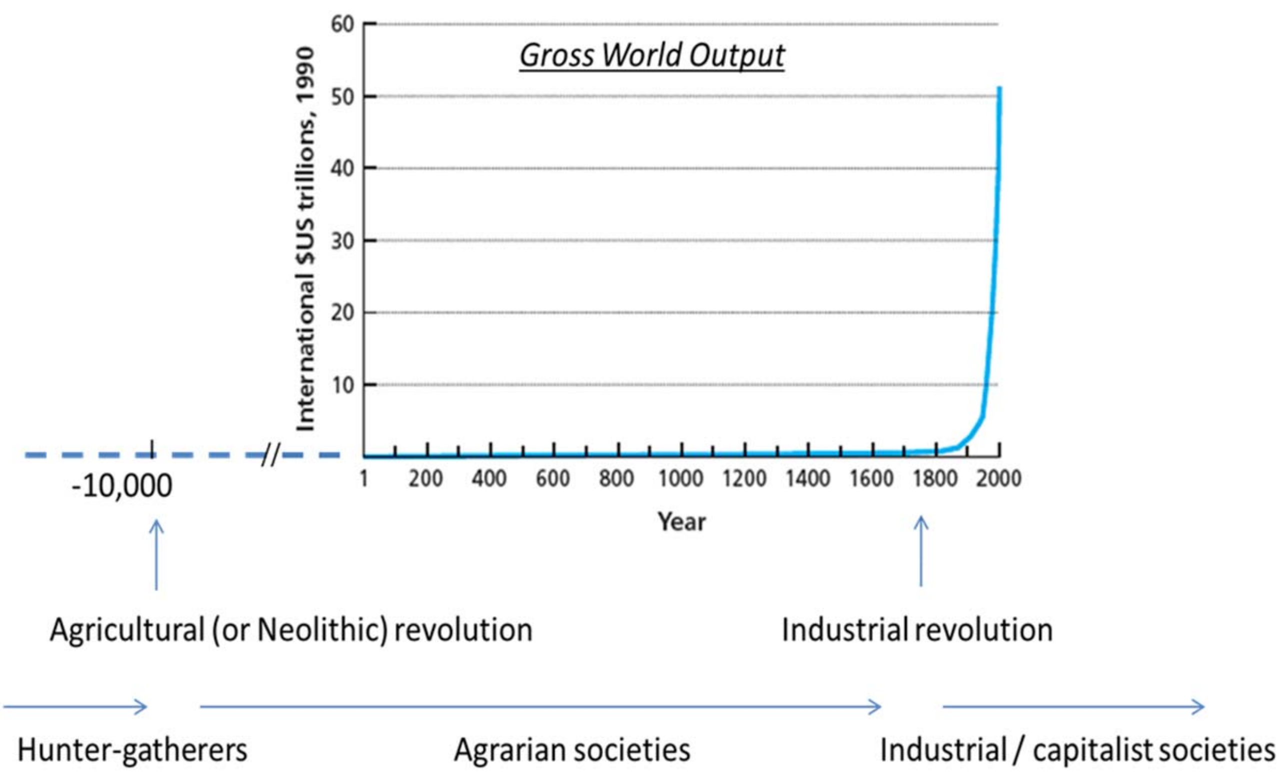
Modeling Companion C
Production Functions

LEARNING OBJECTIVES

- Production functions
- Returns to scale
- Marginal returns

MOTIVATION

The graph below illustrates different estimates of gross world output at different points in time. We know that societies vastly change throughout time. Hunter-gatherers 10,000 years ago, farmers in agrarian societies, and workers from the industrial age evolved in vastly different economic contexts. But how exactly can we model these economies to make their differences more apparent? What are the important aspects of each of these societies that we want our model to reflect? In this process, we shall precisely define production functions for each of these societies.



A. Production Functions

DEFINITION: Production Function 1

A production function is an equation that expresses the relationship between the quantities of productive factors used (such as labor and capital) and the amount of product obtained.

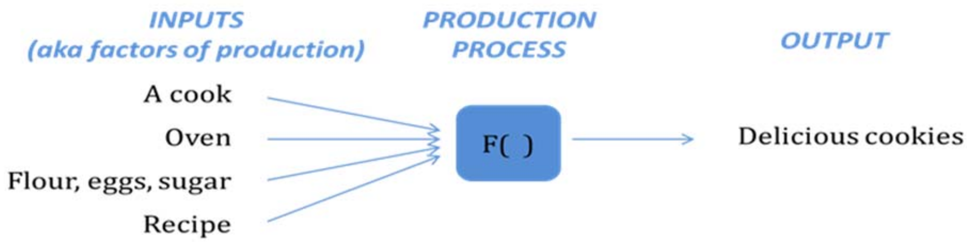
EXAMPLE

The production function of baking cookies - Episode 1

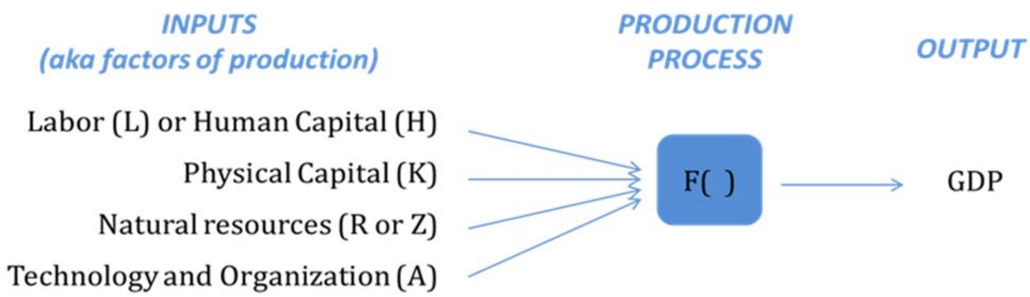
As the figure below shows, to produce cookies you will need several things: a cook (if possible, a good one), an oven, some ingredients like sugar and eggs, and last but not least you need to know what to do. Hence you need a recipe. We can then write that the production function associated with the process of baking cookies is:

$$\boldsymbol{Cookies} = \boldsymbol{F}(\boldsymbol{cook}, \boldsymbol{oven}, \boldsymbol{ingredients}, \boldsymbol{recipe})$$

Here the function F expresses a relationship between the inputs and the output. It specifies exactly how many cookies we can produce given the quantity of inputs we use.



Just like you can produce cookies in your kitchen, the economy of a country produces goods and services that account for total GDP. The figure below shows how a specific production process combines several inputs together to produce output.



There are several factors of production that typically take part in the production processes of a modern economy. We list them below as well as the usual notations that economists like to use.

- Labor (L): number of hours workers supply
- Human Capital (H): education, skills, efficiency
- Physical Capital (K): plants, machinery, equipment
- Natural resources or capital (R or Z): land, commodities such as gold, coal, or timber
- Technology and organization (A): knowledge, techniques, patents, systems of organization, and even sometimes institutions (rules, norms, and culture)

EXAMPLE **The production function of baking cookies - Episode 2**

We can now reexamine our example of baking cookies using the economics jargon. When baking cookies, what you are really doing is using some labor (your own labor, the cook), some physical capital (the oven), some natural resources (eggs, sugar...) and some technique (your grandma's recipe!). Hence we can rewrite the production function of cookies as:

$$\boldsymbol{Cookies} = \boldsymbol{Y} = \boldsymbol{F}(\boldsymbol{L}, \boldsymbol{K}, \boldsymbol{Z}, \boldsymbol{A})$$

B. Properties of production functions

DEFINITION: Returns to Scale

Returns to scale refers to how increasing input in all variables of production affects the resulting output. Suppose we increase the scale of production inputs. By how much will production output increase?

- **Constant returns to scale (CRS):**

When output increases by the same proportional change in inputs

→ **F(aK, aL) = aF(K, L)** (for any constant *a* greater than 0)

- **Decreasing returns to scale (DRS):**

When output increases by less than the proportional change in inputs

→ **F(aK, aL) < aF(K, L)** (for any constant *a* > 1)

- **Increasing returns to scale (IRS):**

When output increases by more than the proportional change in inputs

→ **F(aK, aL) > aF(K, L)** (for any constant *a* > 1)

DEFINITION: Marginal returns

Marginal returns refers to how increasing input in one variable of production, while keeping the others constant, affects the resulting output.

- **Diminishing marginal returns with respect to L:**

each extra unit of input used, holding other types of inputs constant, adds successively less to total production.

Mathematically, this means: $\left(\frac{\partial \mathrm{F}}{\partial \boldsymbol{L}}\right)_{\boldsymbol{K}\ \mathrm{cst}} > 0$ and $\left(\frac{\partial^2 \mathrm{F}}{\partial \boldsymbol{L}^2}\right)_{\boldsymbol{K}\ \mathrm{cst}} < 0$

In most production processes, in the short-run, the quantity of capital is usually fixed, while the quantity of labor can be more easily varied. The law of diminshing marginal returns says that, holding all factors constant except one, beyond some level of the variable input, further increases in the variable input lead to steadily decreasing additions of extra output.

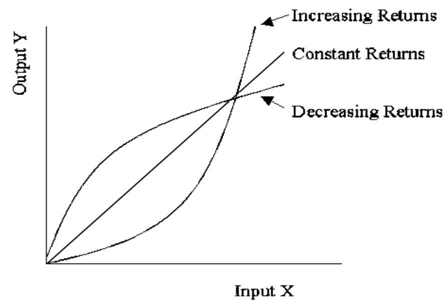
EXAMPLE

Diminishing Marginal Returns in Cookie Making

Think again about our cookie-making activity. Suppose that you have one oven in your kitchen, lots of eggs, sugar, and flour, and the recipe. Say, you are able to cook about 40 cookies an hour because your oven can bake 40 cookies at the same time. Now, suppose you have a second oven. Great! You can bake more cookies; your output increases because you increased your input in ovens. Now say you get a third oven. Well, it's likely that you don't have enough time to make the dough and bake 120 cookies in one hour, so this third oven will not be fully used. Hence, this third oven does not increase your cookie output as much as the second oven did. This is exactly what diminishing marginal returns is about.

GOING FURTHER

Returns (marginal or to scale) have an intuitive graphical representation. Increasing returns can be thought of as a convex curve, decreasing returns as a concave curve, and constant returns as a line.



C. Modeling a hunter-gatherer society

- What should the inputs of the production process be?

Hunter-gatherers survived mostly on collecting fruits and plants from the wild and hunting wild animals. As a result, the key variable that would determine how much food they will get is really the number of hours they will spend gathering or hunting (L), and their technique (A). We can therefore write the production function of a hunter-gatherer society as: $Y = F(A, L)$.

- Will we have decreasing, increasing, or constant returns?

Keeping A constant, if we double the number of people gathering or hunting (2*L), do we expect the amount of food collected to double, less than double, or more than double? With no resource constraint, we should expect the amount of food collected to double. Hence, we will model this society by a linear production function that displays constant returns with respect to labor.

$$\mathrm{Y} = \mathrm{F(A, L)} = \mathrm{A} * \mathrm{L}$$

Given this model of an economy, the only ways to obtain output growth are if either the technologies used improve or if there are more people collecting food.

D. Modeling an agrarian society

- What should the inputs of the production process be?

In agrarian societies, wealth is generated mostly via cultivating the land. As a result, critical inputs appear to be land (Z), farmers working the land (L), and techniques (A). We can therefore write the production function of an agrarian society as: $Y = F(A, L, Z)$.

- Will we have decreasing, increasing, or constant returns?

One farmer cannot work on an ever-increasing number of parcels, and reciprocally, one parcel can not be cultivated by an ever-increasing number of farmers. Hence, we expect decreasing marginal returns with respect to labor and land. In addition, if we increase both the number of farmers and the number of parcels, we should be able to increase food production proportionally. Hence, we expect constant returns when we increase L and Z at the same time. As a result, we will model this society by a Cobb-Douglass production function: $Y = F(A, Z, L) = A\, Z^{\alpha}\, L^{1-\alpha}$, where $0 < \alpha < 1$.

EXAMPLE **Output elasticity of land and labor**

α is also called the output elasticity of land; 1 - α the output elasticity of labor.
If α = 0.45, then:
a 1% increase in land leads to a 0.45% increase in output
a 1% increase in labor leads to a (1 - 0.45)% = 0.55% increase in output
As a result, we have decreasing returns to scale with respect to land and labor.

In this model of an agrarian economy, output can grow by increasing the surface of land that is cultivated, by increasing the number of people working the land, or via technological innovation. Our ancestors might have increased their wealth by increasing the surface of cultivated land, but further increase became soon impossible as most of the fertile land was already occupied. Hence in the agrarian model, once land is considered fixed, the only way to increase wealth is through either technology or labor, but note that with land being fixed, labor displays diminishing returns. Hence, we are really counting on technological innovation to boost agricultural productivity.

E. Modeling an industrial society

- What should the inputs of the production process be?

In industrial societies, wealth is generated mostly via the mass production of goods and services. As a result, critical inputs appear to be technological innovation (A), physical capital (K), and workers (L). We can therefore write the production function of an agrarian society as: $Y = F(A, L, K)$.

- Will we have decreasing, increasing, or constant returns?

One worker cannot work on an ever-increasing number of machines at the same time, and reciprocally, one machine cannot be manipulated by an ever-increasing number of workers. Hence, we expect decreasing marginal returns with respect to labor and capital. In addition, if we increase both the number of workers and the number of machines, we should be able to increase production proportionally. Hence, we expect constant returns when we increase L and K at the same time. As a result, we will model this society by a Cobb-Douglass production function: $Y = F(A, L, K) = A\,K^{\alpha}\,L^{1-\alpha}$, where $0 < \alpha < 1$.

The specification of this production function is very similar to the one in the agrarian society. The major difference is that one has land as an input and the other has physical capital. The key difference here is that land is a fixed factor whereas physical capital can accumulate. As a result, one way to increase wealth is through increasing capital at the same time as we increase labor and technology.

DATA ACTIVITIES

EASY

A. Model 1

Suppose we live in an economy where the production function is linear in inputs, that is:

$$Y = F(A, L) = A * L$$

where Y is the output produced (for example food or any consumption goods), A is the level of technology, and L is labor.

a) Suppose there are 100 farmers in the economy and that A = 2. How many units of output are produced in this economy?

b) How can we have sustained output growth in such an economy? (Which variable would need to grow?)

c) Why is this production function a good model for a hunter-gatherer society?

MEDIUM

B. Model 2

Now suppose we live in an economy where the production function is a Cobb-Douglas function where inputs are technology (A), labor (L), and land (Z). By definition of a Cobb-Douglas function, that is:

$$Y = F(A, Z, L) = A\,Z^{\alpha}\,L^{1-\alpha}$$

Let's assume Z is fixed and cannot grow; Z=100 and $\alpha = 0.45$.

a) Suppose A = 2.
Plot output in this economy for L ranging between 0 and 100. (Choose a sensible interval size to span this range; for example, you can plot for L = 0, 1, 2, 3... or L=0, 5, 10, 15...). Display the graph in your write-up.

b) Does this function display increasing returns to scale?

c) Plot per capita production for L between 0 and 100. Display the graph in your write-up.

d) Describe the graph and explain your intuition.

e) What would happen to income per person if there were a rise in productivity?

f) We are assuming that Z is fixed. Is this a far-fetched assumption?

g) Why is this production function a good model for an agrarian society?

MEDIUM

C. <u>Model 3</u>

Now suppose we live in an economy where the production function is a Cobb-Douglas function where inputs are technology (A), labor (L), and physical capital (K). Y is the output. We also refer to Y as income. By definition of a Cobb-Douglas function, that is:

$$Y = F(A, K, L) = A\,K^{\alpha}\,L^{1-\alpha}$$

We will assume A = 2 and α = 0.45. K and L can vary.

1) In model 2, the amount of land was one of the production inputs, and this amount of land is fixed. In model 3, we use the stock of physical capital as one of the production inputs. Do you think it is reasonable to assume that the stock of physical capital can increase with time?
2) Calculate the output produced in this economy when K = 100 and L = 50.
3) Calculate income per capita.
4) Plot output in this society for L ranging between 0 and 100.
5) If the economy displays constant returns to scale, how much output should be produced when the stock of physical capital is twice as large and when there are twice as many farmers?
6) Does this economy display constant returns to scale? To show this, you can calculate the output produced in this economy when K = 200 and L = 100, and show how it compares to your answer in b).
7) Calculate income per capita. How does it compare to your answer in c)?
8) Show graphically that this production function has constant returns to scale. You can do this by plotting how total production changes when you multiply all the inputs by a varying factor λ, i.e. plot F(λK, λL) against λ holding inputs constant. Why does the graph look like this? What is the slope of the graph? Why?
9) Show graphically that this production function has diminishing marginal returns, i.e. calculate Y for values of L ranging from 0 to 100. Then plot the change in Y (dY) against the population. What does the graph look like and why?
10) For model 2, in question 2c, you looked at the evolution of income per capita when labor increased. Do we have the same trend with model 3? Why or why not?
11) Why is the production function of model 3 a good model for an industrial society?

HARD

D. <u>Model 4</u>

Now use the following production function:

$$\mathrm{Y} = \mathrm{F}(\mathrm{A}, \mathrm{K}, \mathrm{L}) = \mathrm{A}\,\mathrm{K}^{\alpha}\,\mathrm{L}^{\beta}$$

where α= 0.4 and β = 0.5, and all other variables assume the same values as above.

1) Plot output in this society for L ranging between 0 and 100.
2) Generate the same plot as described in question C 8). How do the results differ from those generated with the previous production function? Why?

What if the production function was instead: $Y = 12\,K^{0.7} + 4L$

3) Does this function exhibit constant returns to scale? Does this function generate diminishing marginal returns in all its factor inputs? Justify your answers.